MEM6804 Modeling and Simulation for Logistics and Supply Chain: Theory & Analysis

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

Spring 2021 (full-time)

Assignment 2

Due Date: April 7 (in class)

Instruction

(a) You can answer in English or Chinese or both.
(b) Show enough intermediate steps.
(c) Write your answers independently.

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Question 1 (10 points)

Use Hölder's Inequality to prove that, for real numbers a_i, b_i, $i = 1, 2, \ldots, n$, and positive real numbers p and q such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}.$$

Question 2 (10 points)

Use Minkowski's Inequality to prove that, for real numbers a_i, b_i, $i = 1, 2, \ldots, n$, and real number $p \geq 1$,

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p}.$$

Question 3 (10 points)

For *positive* real numbers a_i, $i = 1, 2, \ldots, n$, define

$$\begin{aligned} \text{arithmetic mean: } a_A &= \frac{1}{n}(a_1 + \cdots + a_n), \\ \text{geometric mean: } a_G &= (a_1 \times \cdots \times a_n)^{1/n}, \\ \text{harmonic mean: } a_H &= \frac{1}{\frac{1}{n}(\frac{1}{a_1} + \cdots + \frac{1}{a_n})}. \end{aligned}$$

Use Jensen's Inequality to prove that $a_H \leq a_G \leq a_A$. (Hint: Use the log() function.)

Question 4 (10 points)

Prove that, if $X_n \xrightarrow{L^s} X$ and $s > r \geq 1$, then $X_n \xrightarrow{L^r} X$. (Hint: Use Hölder's Inequality.)

Question 5 (10 points)

Prove that, if $X_n \xrightarrow{L^r} X$ for $r \geq 1$, then $\mathbb{E}[|X_n|^r] \to \mathbb{E}[|X|^r]$. (Hint: Use Minkowski's Inequality. Also note that for a sequence of numbers $a_1, a_2, \ldots$, and a continuous function $f()$, as $n \to \infty$, $a_n \to a$ implies that $f(a_n) \to f(a)$.)

Question 6 (10 points)

Prove that S^2 is an unbiased estimator of σ^2, i.e., $\mathbb{E}[S^2] = \sigma^2$, but S is a biased estimator of σ, i.e., $\mathbb{E}[S] \neq \sigma$. (Hint: Use Jensen's Inequality.)

Question 7 (10 points)

Prove that $\bar{X}$ is independent of S^2 in the normal distribution case. To make it simple, only consider the case where sample size $n = 2$. (Note: For general case, the analysis is more tedious, but the intuition behind is similar.) (Hint: Try to show that $\bar{X}$ is a function only of $X_1 + X_2$ and S^2 is a function only of $X_2 - X_1$.)

Question 8 (10 points)

Prove that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ in the normal distribution case. [Hint: Let S_n^2 denote the sample variance when the sample size is n. First show that $\frac{S_2^2}{\sigma^2} \sim \chi^2_1$. Then reach the conclusion for $\frac{(n-1)S_n^2}{\sigma^2}$ by induction.]

Question 9 (10 points)

Prove the Weak Law of Large Numbers with iid assumption and $\sigma^2 < \infty$. (Hint: Use Chebyshev's Inequality.)

Question 10 (10 points)

Recall the Numerical Integration example in Lec 1 page 27/32. Let $Y_N = \frac{b-a}{N}[f(X_1) + \cdots + f(X_N)]$. Prove that: (1) $\mathbb{E}[Y_N] = \int_a^b f(x)\mathrm{d}x$; and (2) $Y_N \xrightarrow{a.s.} \int_a^b f(x)\mathrm{d}x$ as $N \to \infty$.